Lecture II: Space–Time Picture of Hadrons

\[ p_1^+ \sim p_z, \quad p_1^- \sim \frac{m_T^2}{p_z} \]

\[ p_2^- \sim p_z, \quad p_2^+ \sim \frac{m_T^2}{p_z} \]

Light Cone Variables:

\[ P^\pm = \frac{1}{\sqrt{2}} (E \pm p_z) \]
\[ X^\pm = \frac{1}{\sqrt{2}} (t \pm z) \]
\[ p^+ p^- = \frac{1}{2} (E^2 - p_z^2) = \frac{1}{2} M_T^2 \]
\[ 2 \ p^+ p^- - p_T^2 = M^2 \]
\[ p \cdot x = p^+ x^- + p^- x^+ - p_T \cdot x_T \]

Conjugate variables:
\[ X^\pm \quad (\pm) \quad p^\mp \]

The uncertainty principle:
\[ \Delta x^\mp \Delta p^\pm \geq 1 \]
For a produced particle, 
\[ x = \frac{p_i^+}{p_1^+} \] is light cone fractional energy 
\[ 0 \leq x \leq 1 \]

It is almost Feynman \( x_F = \frac{E_i}{E_1} \).

Rapidity:
\[ y = \frac{1}{2} \ln\left( \frac{p_i^+}{p_1^+} \right) \sim \ln\left( \frac{2E_i}{M_T} \right) \]

Up to mass effects:
\[ -y_{\text{proj}} \leq y \leq y_{\text{proj}} \]

Spray of particles produced as hadrons pass through one another.

Lorentz time dilation implies fast particles are produced last.
Is There Simple Behaviour at High Energy?

A Hint: Limiting Fragmentation

Renormalization Group

Fast degrees of freedom do not change as $E$ increases.

New degrees of freedom at low energy in c. of m. frame.

New degrees of freedom should be determined by frozen high energy degrees of freedom.

Slow degrees of freedom arise from fast as energy increases.
Physics simple in frame where

\[ P_{\text{hadron}} \to \infty \]

\[ x = \frac{P^+_{\text{constituent}}}{p_{\text{hadron}}^+} \]

Light cone momentum fractions are Bjorken \( x \).

Rapidity:

\[ y = y_{\text{hadron}} - \ln(1/x) \]

\[ x_{\text{min}} \sim \Lambda_{\text{QCD}} / P^+_{\text{hadron}} \]

so that \( y_{\text{min}} \sim 0 \).

Like hadron-hadron scattering, but with \( y \geq 0 \).

Higher energy, add in more degrees of freedom.
Very thin sheet because of Lorentz contraction.

Very large sheet because $p_T \gg 1/R_{had}$

Resolution scale $\Delta x$

is $\Delta x \ll 1/R_{had}$

and $1/\Lambda_{QCD}$.

Physical Picture of a High Energy Hadron

In the limit that $E_{hadron} \to \infty$

for $x \to 0$

$$\Lambda^2 = \frac{1}{\pi R^2} \frac{dN}{dy} \gg \Lambda_{QCD}^2$$

$\alpha_s \ll 1$

$p_T^2 \sim \Lambda^2 \gg 1/R_{had}^2$

The saturation momentum is

$$Q_{sat}^2 \sim \alpha_s \Lambda^2$$

This is probability of interaction with partons at rapidity greater than $y$
Different Rapidity Definitions

Momentum Space Rapidity

\[ y = \frac{1}{2} \ln \left( \frac{p^+}{p^-} \right) = \ln \left( \frac{2p^+}{M_T} \right) = \ln \left( \frac{2p^+_{\text{proj}}}{M_T} \right) + \ln \left( \frac{p^+}{p^+_{\text{proj}}} \right) = y_{\text{proj}} - \ln \left( \frac{1}{x} \right) \]

Coordinate Space Rapidity

\[ y = \frac{1}{2} \ln \left( \frac{x^+}{x^-} \right) = \ln \left( \frac{2\tau}{x} \right) \text{ where} \]

\[ \tau = \sqrt{t^2 - z^2} \]

Using the uncertainty principle

\[ x^+ \sim \frac{1}{p^-} \]

All rapidites same up to \( \Delta y \sim 1 \! \! \! . \)

Can map momentum space into coordinate space!
Distributions of Particles:

Particles with $y_{max} > y > y_{min}$ act as sources for fields with $y_{min} > y$

If the density of sources is big, $\Delta x^2 \rho >> 1$
the sources become \textbf{classical}:

$$[Q^a, Q^b] = i f^{abc} Q_c << Q^2$$

The current associated with this source of color is

$$J_\mu^a = \delta^\mu + \delta(x^-) \rho_a(x_T)$$

The $\delta^{\mu+}$ is because $p^+$ is big.

The $\delta(x^-)$ is approximate and localizes the fields on the sheet $t = z$.

The source $\rho_a(x_T)$ is random in color on 2 dimensional sheet.
The distribution of sources is in reality:
\[ \rho(x^-, x_T) \sim \delta(x^-) \]
where
\[ \rho(x_T) = \int dx^- \rho(x^-, x_T) = \int dy \rho(y, x_T) \]
The width \( \Delta x^- = 1/p^+_{min} \).

Note that \( \rho \) is time, \( x^+ \) independent:

**Glass**

**Yang Mill's theory in presence of random source:**
\[
\int [d\Lambda][d\rho] \exp \left( iS[\Lambda] + iJ^+\Lambda^- - \frac{1}{2} \int dyd^2x_T \frac{\rho^2(y, x_T)}{\mu^2(y)} \right)
\]

The Gaussian ansatz is McLerran-Venugopalan model:
\[
\langle \rho_a(y, x_T)\rho_b(y', x_T) \rangle = \delta^{ab} \delta(y - y')\delta^2(x_T - y_T)\mu^2(y)
\]

\( \mu^2(y) \) is the color charge squared per unit area \( \times y \times Nc^2 - 1 \)

**Random Source \langle \leftarrow \rangle Color Glass \sim Spin**

**Glass**

Incoherent sum \( \langle\leftrightarrow\rangle \) Glass
Some Comments on CGC:

Theory is defined by a cutoff $p_{\text{min}}^+$. The sources arose from fields with $p^+ > p_{\text{min}}^+$. The dynamical fields exist for $p^+ < p_{\text{min}}^+$. Cutoff $p_{\text{min}}^+$ is arbitrary.

Can be changed $\Rightarrow$ Renormalization Group

So long as $p^+/p_{\text{min}}^+$ is not too small, the solution is classical field in presence of $\rho$.

Find solution to

$$D_\mu F^{\mu\nu} = j^\nu$$

and average physical $F[A]$ over $\rho$. Big corrections $\sim \alpha_s \ln(p^+/p_{\text{min}}^+)$ if $p^+/p_{\text{min}}^+$ is too small

$\Rightarrow$ Renormalization Group

Solution to classical equation has $A \sim 1/g$.

The phase space density:

$$\frac{dN}{d^2xdydp_T} \sim <AA> \sim 1/\alpha_s$$

Condensate
The Color Glass Fields

\[
F^{i+} \sim \delta(x^-) \\
F^{i-} \sim 0 \\
F^{ij} \sim 0 \\
F^{i\perp} = F^{i0} \perp F^{iz} \\
E \perp B \perp z
\]

density \sim 1/\alpha_s

fields random

fields frozen in time
The Gluon Distribution and Saturation

Recall that:
\[ \frac{dN}{d^3k} = \frac{2k^+}{(2\pi)^3} A_i^a(k, x^+) A_i^a(-k, x^+) \]
where
\[ A_i^a(x, x^+) = \frac{i}{q} U(x) \nabla_i U^†(x) \]

You can compute:

\[ \langle A_i^a(x, x^+) A_i^a(y, x^+) \rangle = \frac{N_c^2 - 1}{\pi \alpha_s N_c} \left( 1 - e^{\frac{x_T^2 Q_s^2 ln(x_T^2 \Lambda_{QCD}^2)}{4}} \right) \]

Also
\[ \int dx^- \mu^2(x^-) = \int_{y_{min}}^{y_{max}} dy \mu^2(y) \]
so it is the total charge at all rapidities greater than where we measure. This can related to the gluon density by DGLAP and that charge density, up to Casimir is gluon density.

In this equation, both \( x^- \) and \( y^- \) are outside the range where the source sits. The saturation momentum is:

\[ Q_s^2 = 2\pi N_c \alpha_s^2 \int dx^- \mu^2(x^-) \sim \alpha_s^2 \frac{charge^2}{area(x_i (N_c^2 - 1))} \]

Formula true only for \( x_T << 1/\Lambda_{QCD} \)
Hadron Collisions

The fields before the collision:

Nucleus $1: F^{i+} \sim \delta(x^-), F^{ij} \sim 0, F^{i-} \sim 0$.

Nucleus $1: F^{i+} \sim \delta(x^+), F^{ij} \sim 0, F^{i+} \sim 0$.

Plane polarized and random color.
Fields are $2-d$ gauge transforms of zero field everywhere but in the forward lightcone.

In the forward lightcone matter is produced.
No solution with gauge transform of vacuum in all light cones.
Choose $A = 0$ in backward light cone.

In left and right halves, pure gauge.

Discontinuity across light cone to match color charge sources on light cone

Field is not pure gauge in forward lightcone

Physical motivation: Renormalization group description.

In center of mass frame, degrees of freedom with

$$y \ll \frac{1}{\alpha_S}$$

are coherent fields.

Larger $y$ are sources

$$\alpha_S(Q_S) \ll 1$$
Before the collision, two sheets of mutually transverse color electric and color magnetic fields.

**Boosted Coulomb fields**

**Random in color**

**Thickness of sheets is**

\[ \Delta z \sim \frac{1}{Q_S} e^{-\kappa/\alpha_S} \]
Initial fields:

\[ \alpha^i_{(1,2)} = \frac{1}{ig} U_{(1,2)}(x_T) \nabla^i U_{(1,2)}^\dagger(x_T) \]

In radial gauge,

\[ x^+ A^- + x^- A^+ = 0 \]

the fields in the forward light cone are:

\[ A^\pm = \pm x^\pm \alpha(\tau, x_T) \]

\[ A^i = \alpha^i_3(\tau, x_T) \]

Assume boost invariant solution
Boundary conditions are determined by solving equations across the light cone:

Infinitesimally after the collision there are

No transverse fields

Longitudinal magnetic and electric fields

\[ E^z = ig[\alpha_1^i, \alpha_2^i] \]

\[ B^z = ig\epsilon_{ij} [\alpha_1^i, \alpha_2^j] \]
These fields have a local topological charge density

Chern-Simons charge

\[ FF^d \sim \partial_\mu K^\mu \]

The Chern-Simons charge density is maximal!

\[ FF^d \sim 1/g^2 \]

and has a transverse correlation length

\[ \Delta x_T \sim 1/Q_S \]
How do the sources of color magnetic and color electric field arise?

\[
D \cdot E = 0
\]

\[
\nabla \cdot E = -g[A, E]
\]

\[
D \cdot B = 0
\]

\[
\nabla \cdot B = -g[A, B]
\]

In forward light cone, the vector potential from one nucleus can multiply the CGC field from the other.

Equal and opposite densities of charge
However:

Glasma fields are initial conditions, not a solution to time independent equation of motion:

$$D_\mu F^{\mu\nu} = 0$$

$$D_0 \vec{E} = \vec{D} \times \vec{B}$$

Unlike the constant field where there is no magnetic field:

$$\frac{d}{dt} \vec{E} = 0$$
The Lund model made the daring proposal that there were longitudinal electric fields which decay by pair production.

There is also a longitudinal magnetic field.

It can also decay by rearrangement of the charge in the classical field (classical screening) which is naively dominant.

Kharzeev and Tuchin and Janik, Shuryak and Zahed made the daring proposal that particles are made by decay of Chern-Simons charge.

Both are correct!

They are included in the color glass initial conditions!

The matter which is this melting glass, or hadronizing strings or sphaleron decays is the Glasma.
The Glasma has three components:

Coherent classical fields:

Hard particles:

Degrees of freedom which can be described as either hard particles or coherent fields

\[ A \sim \frac{1}{g} \quad p_T \ll \frac{1}{\tau} \]

\[ A \ll \frac{1}{g} \quad p_T \gg \frac{1}{(\alpha_S \tau)} \]

\[ 1 \ll A \ll \frac{1}{g} \quad \frac{1}{\tau} \ll p_T \ll \frac{1}{(\alpha_S \tau)} \]

The Glasma has mostly evaporated by a time \( \tau \sim \frac{1}{(\alpha_S Q_S)} \)

During this time, scattering among the hard modes (parton cascade) is not important.
Interactions in the coherent fields takes place on a scale of order $1/Q_s$

Because of coherence, interactions of hard particles with the classical fields,

$$g \times 1/g \sim 1$$

Also take place on a time scale $1/Q_s$

Very rapid strongly interacting system

But boost invariance is a problem, as this does not allow longitudinal momentum to become thermalized

Important for two reasons:

Almost certainly instabilities of the hard-soft coupled system under boost non-invariant perturbations

The local topological charge wants to decay, and this is easiest with a boost non-invariant distribution
Consequence of nonzero Chern-Simons Charge: Vorticity Generation

\[ E \cdot B \neq 0 \]

Positively charged particle accelerates along \( E \),
rotates in clockwise direction

Negatively charged particle accelerates along \(-E\),
Rotates in anticlockwise direction
Net vorticity generation

Physical origin of t Hooft anomaly
The total cross section is infrared sensitive:

$$\frac{d\sigma}{dy} \sim \alpha_s^2 \int_{\Lambda_{QCD}} \frac{d^2p_T}{p_T^4}$$

Will argue cutoff at small $p_T$ is $Q_{sat}$:

$$\frac{1}{\pi R^2} \frac{dN}{d^2p_T dy} = \kappa \frac{1}{\alpha_s \ln(Q_{sat}^2/\Lambda_{QCD}^2)} \frac{1}{p_T^4}$$

is cutoff in the infrared.

If:
Physics is classical fields $\Rightarrow$ then scale invariance $\Rightarrow$

$$\frac{1}{\pi R^2} \frac{dN}{d^2p_T dy} = \frac{1}{\alpha_s} F(Q_{sat}^2/p_T^2)$$

where

$F \sim Q_{sat}^4/p_T^4$ for large $p_T \gg Q_{sat}$
and $F \sim \text{constant}$ for $p_T \ll Q_{sat}$.

Recall that $Q_{sat}^2 \sim A^{1/3}$

$$\frac{dN}{d^2p_T dy} \sim \pi R^2 \frac{Q_{sat}^4}{p_T^4} \sim A^{4/3}/p_T^4$$

at large $p_T$. At small $p_T$, $\sim \pi R^2 \sim A^{2/3}$.

$$\int d^2p_T \frac{dN}{d^2p_T dy} \sim \pi R^2 Q_{sat}^2/\alpha_s \sim A/\alpha_s.$$
Numerical Results for Gluon Production

Numerical results similar to Bose-Einstein distribution at small $p_T$

$1/p_T^4$ at large $p_T$

Proportional to $1/\alpha_s$
Thermalization:

Slow particles form first. Fast are last.

Like Hubble expansion.

\[ V \sim \frac{1}{\tau} \]

Entropy conserved:

\[ S \sim T^3 \tau R^2 \]
\[ T \sim \tau^{-1/3} \]

Energy density \( \sim T^4 \) for thermal \( \epsilon \sim \tau^{-4/3} \)
For non thermal \( \epsilon \sim \tau^{-1} \)
At late times when energy per particle is slowly varying measure

\[ \frac{dE}{dy} = \epsilon \tau \pi R^2 \]

Can measure \(<m_T>\) and \(dN/dy\).
Multiplicity and Energy Density

\[ \frac{dN_{\text{ch}}}{dN_{\text{ch}}} |_{n<1} \text{ vs Energy} \]

\[ \text{dN/dN}(0.5 \text{ N}_{\text{part}}) \]

\[ s^{1/2} \text{ (GeV)} \]

Melting Colored Glass
Energy Density at Formation \( \varepsilon \sim 20-30 \text{ GeV/Fm}^3 \)
Quark Gluon Matter
Quark Gluon Plasma
Bjorken Energy Density \( \varepsilon \sim 2-3 \text{ GeV/Fm}^3 \)
Mixed Hadron Gas and Quark Gluon Plasma

Energy Density
- ~20-30 times that inside a proton
- in Cores of Neutron Stars
- Nuclear Matter

nucl-ex/0108009
Submitted to PRL
Late Stages

Color glass $\rightarrow$
Quark-Gluon Matter $\rightarrow$
Quark-Gluon Plasma $\rightarrow$
Mixed phase $\rightarrow$
Hadron Gas $\rightarrow$
Free Streaming Hadrons.

Classical Fields
Classical Fields $+$ Transport?
Hydrodynamics
Hydrodynamics
Hydrodynamics
Deoupling algorithm or cascade.